LETTER TO THE EDITOR

Observables and gauge invariance in the theory of non-linear spacetime perturbations

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Abstract. We discuss the issue of observables in general-relativistic perturbation theory, adopting the view that any observable in general relativity is represented by a scalar field on spacetime. In the context of perturbation theory, an observable is therefore a scalar field on the perturbed spacetime, and as such is gauge invariant in an exact sense (to all orders), as one would expect. However, perturbations are usually represented by fields on the background spacetime, and expanded at different orders into contributions that may or may not be gauge independent. We show that perturbations of scalar quantities are observable if they are first order gauge-invariant, even if they are gauge dependent at higher order. Gauge invariance to first order plays therefore an important conceptual role in the theory, for it selects the perturbations with direct physical meaning from those having only a mathematical status. The so-called "gauge problem", and the relationship between measured fluctuations and gauge dependent perturbations that are computed in the theory are also clarified.

The issue of what is observable in general relativity is still a controversial one[1]. Here we want to address a related problem, which is fundamental for practical purposes: which *perturbations* are observable in general relativity? Clearly, the answer to this question is largely conditioned on one's attitude towards the more basic issue mentioned above. However, we believe that this question deserves an analysis of its own, given the peculiar nature of gauge issues in the context of relativistic perturbation theory, and the fact that the comparison of Einstein's theory with observations is almost entirely based on approximation methods (see e.g. [2] for a discussion of this point).

In the following, we adopt the view that an observable quantity in general relativity¶ is simply represented by a scalar field on spacetime. This definition may seem naive, but it corresponds exactly to what most physicists have in mind when thinking about observable quantities. At a more sophisticated level, it can be argued that such a notion of observables is practically viable even if one considers the issues related to the invariance of general relativity under spacetime diffeomorphisms.

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¶ Our discussion is however valid in any spacetime theory.

In the relativistic theory of perturbations one is always dealing with two spacetimes, the physical (perturbed) one, and an idealised (unperturbed) background. In this context, given the above definition, an observable is a scalar on the perturbed spacetime. Thus, physical quantities such as, e.g., the energy density in a cosmological model, trivially satisfy our definition of observables. However, in perturbation theory one formally decomposes quantities of the physical space into the sum of a background quantity and a perturbation. The question arises therefore, whether perturbations themselves can be regarded as observables. Indeed, it is certainly useful to express the perturbative formalism directly in terms of variables (the perturbations) that are observable quantities, or at least to have clear in mind how to relate mathematical variables with physically meaningful ones. Given the recent development of second order perturbation theory in cosmology (see [3, 4] and references therein) and black hole physics (see [5, 6] and references therein), the issue becomes even more important, because of the further complications that arise when non-linearities are considered.

As we shall discuss shortly, the perturbations are represented by fields on the background, and expanded at different orders into contributions that may or may not be gauge independent. On the other hand, one would expect that observables are described by gauge-invariant quantities. Recently, explicit calculations have shown that fields which are usually regarded as representing observables are not gauge-invariant at second order[3, 4, 6], although it has long been known that they are gauge-invariant at first order (see e.g. [7, 8, 9]). In this letter we point out that the perturbation of a scalar describes an observable if and only if its representation on the background is gauge-invariant at first order, even when it is gauge dependent at higher orders. We shall also discuss how this seemingly paradoxical result corresponds to have observable perturbations that are gauge-invariant in the exact sense (i.e., at all orders) in the physical spacetime, as expected. Finally, we shall briefly comment on how a first order gauge dependent perturbation can acquire physical meaning in a specific gauge.

Let us begin by reviewing some general ideas about the perturbative approach in general relativity.[+] Suppose that the physical and the background spacetimes are represented by the Lorentzian manifolds $(\mathcal{M}, g)$ and $(\mathcal{M}_0, g_0)$, respectively (as manifolds, $\mathcal{M}_0 = \mathcal{M}$, but it is nevertheless convenient to label them differently). The perturbation of a quantity should obviously be defined as the difference between the values that the quantity takes in $\mathcal{M}$ and $\mathcal{M}_0$, evaluated at points which correspond to the same physical event. However, there is nothing intrinsic to $(\mathcal{M}, g)$ and $(\mathcal{M}_0, g_0)$ that allows us to establish a one-to-one correspondence between the two manifolds. This follows directly from the fact that no spacetime structure is assigned *a priori* in general relativity, contrarily to what happens, e.g., in the Newtonian theory, where $\mathcal{M}_0$ and $\mathcal{M}$ are simply identified, thus making possible a straightforward formulation of Eulerian perturbation theory [13] (see, however, [14] and references therein for a Lagrangian formulation in Newtonian cosmology). Hence, in general relativity, points of $\mathcal{M}$ are unrelated with points of $\mathcal{M}_0$, and if we want to compare a physical quantity in the two spacetimes, we need an additional prescription about the pairwise identification of points between $\mathcal{M}$ and $\mathcal{M}_0$. Mathematically, this corresponds to the assignment of a diffeomorphism $\varphi : \mathcal{M}_0 \to \mathcal{M}$, sometimes called a *point identification map* [8]. Such a diffeomorphism can be given directly as a mapping between $\mathcal{M}_0$

[+] Hereafter, we shall adopt the conventions and notations of references [10, 11]. See also reference [12] for the more general mathematical definitions.

and $\mathcal{M}$ or — perhaps more commonly — by choosing charts X in $\mathcal{M}_0$ and Y in $\mathcal{M}$ (with coordinates $\{x^\mu\}$ and $\{y^\mu\}$, respectively), and identifying points having the same value of the coordinates [2, 15], so that φ is implicitly defined through the relation $x^\mu(p) = y^\mu(\varphi(p))$, $\forall p \in \mathcal{M}_0$. In any case, the point identification map is completely arbitrary; this freedom is peculiar of general relativistic perturbation theory, and has no counterpart in theories that are formulated on a fixed background, where no ambiguity arises in comparing fields. Following Sachs [16, p 556], one may refer to it as gauge freedom "of the second kind", in order to distinguish it from the usual gauge freedom of general relativity. However, in the following we shall never consider ordinary gauge transformations of the perturbed and background spacetimes, and we shall therefore use the term "gauge" as synonym of "gauge of the second kind".

Once a gauge choice has been made (i.e., a point identification map φ has been assigned), perturbations can be defined unambiguously. Let T be a tensor field on $\mathcal{M}$. If T_0 is the background tensor field corresponding to T, the total *perturbation* of T is simply given by $\Delta^\varphi T := T - \varphi_* T_0$ [10, 11]. By definition, $\Delta^\varphi T$ is a field on $\mathcal{M}$. On the other hand, the aim of perturbation theory in general relativity is to *construct*, through an iterative scheme, the geometry on $\mathcal{M}$ (see e.g. [17]). To this purpose, it is customary to work with fields on $\mathcal{M}_0$ (see [10, 11] and references therein). Thus, to start with, the *representation* on $\mathcal{M}_0$ of an arbitrary tensor field T is defined as the pull-back $\varphi^* T$. Then, the representation on the background of the perturbation is $\Delta_0^\varphi T := \varphi^* \Delta^\varphi T = \varphi^* T - T_0$. These are the background fields that are Taylor-expanded to obtain the contributions at different orders that are used in the above mentioned iteration scheme.

We would like to point out here that, for practical purposes, it is often very convenient to use, on $\mathcal{M}_0$ and $\mathcal{M}$, coordinates $\{x^\mu\}$ and $\{y^\mu\}$ "adapted" to φ, such that $x^\mu(p) = y^\mu(\varphi(p))$, $\forall p \in \mathcal{M}_0$. In this way the components of T at the point $\varphi(p) \in \mathcal{M}$ coincide with those of $\varphi^* T$ at $p \in \mathcal{M}_0$ (see figure 1 for a pictorial explanation in the case of a vector). Obviously, the same is true for $\varphi_* T_0$ and T_0, so that one has, for a tensor of type (r, s),

$$(\Delta^\varphi T)^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s}(x) = (\Delta_0^\varphi T)^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s}(x) = T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s}(x) - T_0^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s}(x) , \tag{1}$$

where $x = X(p) = Y(\varphi(p))$. This choice is however rather confusing from a conceptual point of view because, once the above identification has been made, it is hard to distinguish between $\Delta^\varphi T$ and $\Delta_0^\varphi T$. The point is that, as we shall show, $\Delta^\varphi T$ may be gauge-invariant in an exact sense (at all orders) and thus, if T is a scalar, it may correspond to an observable, even when $\Delta_0^\varphi T$ is gauge-invariant only at first order.

Under a gauge transformation $\varphi \to \psi$, where ψ is another point identification map, the representation on $\mathcal{M}_0$ of tensor fields defined on $\mathcal{M}$ changes as under the action of a diffeomorphism. This can easily be seen by noticing that a point $p \in \mathcal{M}$ corresponds, in two gauges φ and ψ, to the points $\varphi^{-1}(p)$ and $\psi^{-1}(p)$ in $\mathcal{M}_0$. Defining a map $\Phi : \mathcal{M}_0 \to \mathcal{M}_0$ as $\Phi := \varphi^{-1} \circ \psi$, we have $\varphi^{-1}(p) = \Phi(\psi^{-1}(p))$. Then the two representations on $\mathcal{M}_0$ of a tensor T of $\mathcal{M}$ are related by $\psi^* T = (\varphi \circ \Phi)^* T = \Phi^*(\varphi^* T)$. It follows that the gauge transformation between the two representations of the perturbation of T in the two gauges is:

$$\Delta_0^\psi T = \Phi^* \Delta_0^\varphi T + \Phi^* T_0 - T_0 = \Phi^*(\varphi^* T) - T_0. \tag{2}$$

We want to point out here that from the first equality in this equation it would seem that a fundamental fact about gauge transformations is that Φ^* acts on T_0. However,
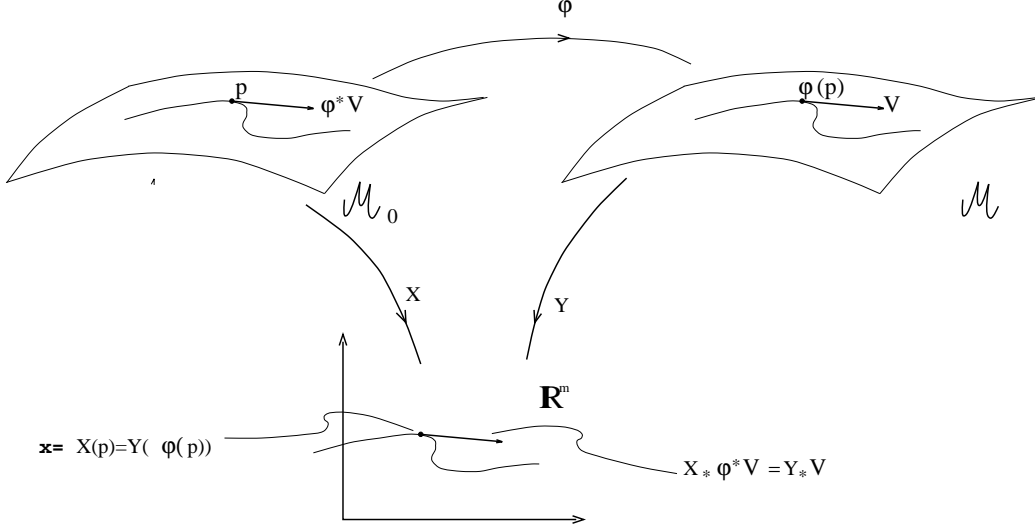
Figure 1. By choosing the coordinates on $\mathcal{M}_0$ and $\mathcal{M}$ in such a way that $y^\mu \circ \varphi = x^\mu$, a curve in $\mathcal{M}_0$ and its φ-transformed in $\mathcal{M}$ have the same representation in $\mathbb{R}^m$. Therefore, the components of the tangent vectors V and $\varphi^* V$ at the points $\varphi(p)$ and p are the same: $(\varphi^* V)^\mu (x) = (\varphi^* V)(x^\mu)|_p = V\left(x^\mu \circ \varphi^{-1}\right)|_{\varphi(p)} = V(y^\mu)|_{\varphi(p)} = V^\mu(x)$.

it is clear from the second equality that this is not the case, as all is really needed is the action of Φ^* on the representation on $\mathcal{M}_0$ of the field T of the physical spacetime. In other words, it is only the fact that we insist to look at the gauge transformation of $\Delta_0^\varphi T$ as a whole that brings about the $\Phi^* T_0$ term in the first equality. In this sense, it would be an improper extrapolation to say that "a gauge transformation is equivalent to a diffeomorphism of the background".

It is useful to classify fields on $\mathcal{M}$, according to whether they are *intrinsically gauge independent* (IGI) or *intrinsically gauge dependent* (IGD). We say that a field is IGI iff its value at any point of $\mathcal{M}$ does not depend on the gauge choice; otherwise, we say that it is IGD. As obvious examples of IGI and IGD quantities we mention, respectively, a tensor field T defined on $\mathcal{M}$, and the push-forward $\varphi_* T_0$ on $\mathcal{M}$ of a non-trivial* tensor field T_0 defined on $\mathcal{M}_0$. It follows that, given our identification of observables with scalar functions on $\mathcal{M}$ and the arbitrariness in the choice of the gauge φ, a scalar describes an observable only if it is IGI. On the other hand, considering the representation on $\mathcal{M}_0$ of the perturbations, one is led to define a corresponding idea in the background, saying that a quantity on $\mathcal{M}_0$ is *identification gauge-invariant* (i.g.i.) iff its value at any point of $\mathcal{M}_0$ does not depend on the gauge choice [8]. An example of i.g.i. quantity is a tensor field T_0 defined on $\mathcal{M}_0$, while the pull-back $\varphi^* T$ of a tensor field T defined on $\mathcal{M}$ is not i.g.i. unless T is trivial. It is then clear that the representation on the background of an IGI quantity is not i.g.i. in general, but this is totally irrelevant as far as the issue of observability is concerned, because measurements are always performed in the physical spacetime $\mathcal{M}$, whereas the background $\mathcal{M}_0$ has merely the status of a useful mathematical artifice.

* Hereafter by a trivial tensor we mean one that is either vanishing or a constant multiple of the identity [8].

Let us now turn our attention to perturbations, asking whether they are IGI or IGD, and how this relates to the gauge dependence of their representation on $\mathcal{M}_0$. In order to answer these questions, we consider a gauge transformation $\varphi \to \psi$. Correspondingly, perturbations transform as $\Delta^\varphi T \to \Delta^\psi T$, where

$$\Delta^\psi T = \Delta^\varphi T + (\varphi_* T_0 - \psi_* T_0) \ . \tag{3}$$

Similarly, their representations on the background transform as $\Delta_0^\varphi T \to \Delta_0^\psi T$, with

$$\Delta_0^\psi T = \Delta_0^\varphi T + (\psi^* T - \varphi^* T) \ . \tag{4}$$

Therefore, in general, both the perturbations on $\mathcal{M}$ and their representations on $\mathcal{M}_0$ change under the action of a gauge transformation. This gauge dependence of perturbations does not appear in theories that admit a canonical identification between $\mathcal{M}$ and $\mathcal{M}_0$, and is due to the arbitrariness in the choice of a point identification map, which is additional to the usual gauge freedom of general relativity. In a sense, general relativistic perturbations "have gauge freedom of their own", i.e. the freedom "of the second kind"[16] mentioned before, even when the full quantities have not. We have to remind at this point that, once the Taylor expansion into different order contributions has been carried out, it turns out that a perturbation on $\mathcal{M}_0$ may be gauge-invariant at first order, and not at higher orders [10, 11]. Now, it follows from (3) that the perturbation is IGI iff $\varphi_* T_0 = \psi_* T_0$, $\forall \varphi, \psi$. This can be rewritten as $T_0 = \Phi^* T_0$, where $\Phi := \varphi^{-1} \circ \psi : \mathcal{M}_0 \to \mathcal{M}_0$, and is satisfied only if T_0 is trivial. But this is precisely the condition for first order i.g.i. derived by Stewart and Walker [8]. Thus, the perturbation of T is IGI iff its representation on the background is gauge-invariant to first order. For the particular case in which T is a scalar physical quantity, we obtain the main result of this letter: the perturbation of T is observable iff its representation on $\mathcal{M}_0$ is first order i.g.i., even when it is gauge dependent to higher orders.

This result may sound trivial, but only because we have approached the question of observability of perturbations from the side of the physical spacetime $\mathcal{M}$. Focusing attention only on perturbations as fields on $\mathcal{M}_0$ rather than on $\mathcal{M}$, as it is usually done in the iterative scheme, the notion of IGI quantities does not naturally arise, and one would ask instead whether the representation of a perturbation is i.g.i., i.e., whether $\Delta_0^\varphi T = \Delta_0^\psi T$, $\forall \varphi, \psi$ identification maps [8]. Because of (4), this condition is equivalent to the requirement that $\varphi^* T = \psi^* T$, $\forall \varphi, \psi$, which is satisfied iff T is trivial. Thus, the perturbation of a quantity is IGI iff the quantity itself is trivial in the background $\mathcal{M}_0$, while its representation is i.g.i. iff the quantity is trivial on $\mathcal{M}$. The important point is that the physically interesting condition is not i.g.i., but IGI, as one can clearly see by considering the case in which T is a scalar. The requirement that the perturbation of T be i.g.i. amounts to saying that T must be a constant on $\mathcal{M}$, which is too strong a constraint to be fulfilled by most observables of physical interest. On the contrary, IGI requires only that T_0 be constant. As we have already pointed out, the physical content of the theory resides in the quantities on $\mathcal{M}$, not in their representations on $\mathcal{M}_0$. From the physical point of view, there is nothing bad in having a gauge dependent pull-back on $\mathcal{M}_0$, provided that the quantity on $\mathcal{M}$ be uniquely defined. Of course, to the first order in a perturbative approach, the perturbation of a quantity is IGI iff its representation is i.g.i.

The gauge dependence of perturbations has often been referred to in the past as the *gauge problem* [9, 15].♯ It is now clear that, for IGI perturbations, such gauge dependence is only an artifact of focusing attention on the representation, and

♯ A secondary issue that follows from this gauge dependence is that of gauge modes. The latter

there is no real problem in the physical spacetime $\mathcal{M}$. The situation is different for IGD perturbations. Their very definition requires a gauge to be defined: this *is* a difficulty, because then these quantities cannot directly represent observables. However, especially in cosmology, it is often the case that astronomers *do* measure quantities that they call perturbations, or fluctuations, which they define with respect to *an averaged quantity*. It is obvious that such fluctuations must be IGI, as their very definition does not involve any background or gauge choice. A paradigmatic example is that of the Cosmic Microwave Background (CMB) temperature anisotropy (see [19, 20] and references therein). We are then facing a paradox: on the one hand theoretical perturbations that are commonly considered are IGD (see e.g. [9]), whereas their observational counterpart are IGI. It is then necessary to understand what is the relationship between them.

The resolution of the problem lies in recognising the misleading identification that is often made between averages in the physical spacetime and background quantities. Supposing that an averaging procedure has been adopted,†† the measured fluctuation of a quantity T is defined by the observer simply as $\Delta T := T - \langle T \rangle$. Clearly, such a definition is valid in any spacetime and, by itself, has no relation whatsoever with the adoption of a background model and/or of a perturbative formalism (of which the observer can be happily totally unaware). However, if we want to consider this spacetime from a perturbative point of view, then such a relation is easily established by rewriting ΔT as

$$\Delta T = T - \langle T \rangle = (T - \varphi_* T_0) - (\langle T \rangle - \varphi_* T_0) = \Delta^\varphi T - \Delta^\varphi \langle T \rangle , \quad (5)$$

where the same background quantity T_0 has been used for both T and $\langle T \rangle$. Thus, the theoretically computed perturbation, $\Delta^\varphi T$, differs from the measured one, ΔT, by the term $\Delta^\varphi \langle T \rangle$. The apparent paradox outlined above arises therefore when $\Delta^\varphi T$ is identified with ΔT, which happens if one thinks of the average $\langle T \rangle$ as the push-forward $\varphi_* T_0$ of some T_0 defined in the background spacetime. This is generally wrong, as one can see by considering the example of the CMB anisotropy, where $\langle T \rangle$ is obtained by an angular average [19, 20] and depends therefore on the spatial position as well as on time, while T_0 can depend only on time, being the temperature in a spatially homogeneous cosmological model. The only case in which the identification is meaningful is when $\langle T \rangle$ and T_0 are constant on spacetime. Indeed, $\langle T \rangle$ is IGI by definition, while in general $\varphi_* T_0$ is IGD. If we want $\langle T \rangle = \varphi_* T_0$, it is clear that $\varphi_* T_0$ must also be IGI, which can be the case only if T_0 is a constant. Instead, of particular interest in cosmology is the case when $\langle T \rangle_0 = T_0$, i.e. when we consider a sky or spatial average, so that ΔT has a vanishing background value, and therefore is IGI (and, of course, first order i.g.i.).

Scalars that deserve specific attention are those built by projecting a tensor over a tetrad: a well-known example is that of the Weyl scalars in the Newman-Penrose (NP) formalism, which are often used in the study of black hole perturbations [6, 7, 8]. Let us consider, in order to fix the ideas, the NP scalar $\Psi_4 := C(l, \overline{m}, l, \overline{m})$, where C

arise when the prescription of point identification has not been given, or when it is not sharp enough to select a single $\varphi : \mathcal{M}_0 \to \mathcal{M}$, and defines instead a *class* of diffeomorphisms [18]. Of course, for gauge-invariant perturbations both problems disappear.

†† Such an average can be performed on data taken on suitable subspaces of the whole spacetime. A particularly interesting case is that of averages on the sky made by each observer at his spacetime point. Obviously, in general a sky-averaged quantity is point-dependent, an important fact for the following discussion.

is the Weyl tensor and $\{l, n, m, \overline{m}\}$ is the usual NP null tetrad. The perturbation of Ψ_4 is

$$\Delta^\varphi \Psi_4 = \Psi_4 - \varphi_* \Psi_{4,0} \, , \tag{6}$$

where $\Psi_{4,0} = C_0(l_0, \overline{m}_0, l_0, \overline{m}_0)$ is constructed only from quantities defined on $\mathcal{M}_0$. It may seem that $\Delta^\varphi \Psi_4$ contains more arbitrariness than the perturbations we have considered so far, because its definition requires not only to choose a gauge φ, but also to specify a tetrad $\{l_0, n_0, m_0, \overline{m}_0\}$ in the background and another one, $\{l, n, m, \overline{m}\}$, in the physical space. Actually, the two tetrads are not independent, because if we require that $\Psi_4 \to \Psi_{4,0}$ in the limit of vanishing perturbations, we must have, e.g., $l = \varphi_* l_0 + \Delta^\varphi l$, so that the perturbed tetrad can be constructed iteratively from the unperturbed one, by imposing that it be null with respect to the perturbed metric, order by order. Nevertheless, we are still left with a dependence of $\Delta^\varphi \Psi_4$ on the background tetrad. However, it is easy to understand that this tetrad dependence is not a problem as far as observability is concerned. Indeed, changing the tetrad from $\{l_0, n_0, m_0, \overline{m}_0\}$ to $\{l_0', n_0', m_0', \overline{m}_0'\}$, say, Ψ_4 will change to another scalar Ψ_4' with a *different* physical/geometrical interpretation. Tetrad dependence corresponds thus to the possibility of constructing different measurable quantities starting from the Weyl tensor, and has not to be regarded, conceptually, on the same footing of the gauge dependence that we have considered in the rest of this letter. Also, in practice one is guided by the Petrov algebraic type of the background in choosing on it a specific tetrad, such that some of the Weyl scalars vanish and thus are first order i.g.i. For example, Schwarzschild and Kerr spacetimes are Petrov type D, and the tetrad can be aligned with the principal null directions, so that only $\Psi_2 \neq 0$, which results in Ψ_0 and Ψ_4 describing gravitational wave perturbations in a gauge-invariant way [6, 7, 8]. At second order, the construction of a gauge *and* tetrad invariant NP perturbative formalism is computationally very useful, e.g. in order to compare results with those of a fully numerical treatment. For Kerr, this can be achieved [6] by implementing a procedure that strongly reminds the one used at first order in cosmology [15]. One can construct infinitely many of such gauge-invariant quantities, but again physical meaning guides the choice: in [6] the chosen second-order i.g.i. quantity is the one reducing to Ψ_4 in an asymptotically flat gauge.

Finally, one may wonder what gauge dependent variables, which are commonly used in perturbation theory, have to do with observables. The answer to this question is obvious: provided that calculations are free from gauge modes, even a first order gauge dependent perturbation acquires physical meaning in a specific gauge when, in that gauge, it can be identified with a gauge-invariant quantity. This is the case, for example, of the density perturbation $\delta\rho/\rho$ in cosmology: its value in the comoving gauge coincides with the value in that gauge of one of Bardeen's gauge-invariant variables [15], and in turn the latter is the first order expansion of a covariantly defined density gradient [21]. It is our opinion that there is more appeal in working directly with observable variables which are automatically IGI, such as the fluctuation ΔT considered above. However, it may often be the case that one is not able to find a complete set of such quantities, so that one will either resort to gauge-dependent variables, or to appropriate combinations of gauge-dependent quantities that are gauge-invariant at the desidered order [6, 15].

Acknowledgments

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